Advanced Mechanics: Resit

Semester Ib 2021-2022

April 13th, 2022

Please note the following rules:

- you are not allowed to use the book or the lecture notes, nor other notes or books
- you have 2 hours to complete the exam
- please write your student number on each paper sheet you hand in
- please raise your hand for more paper or to ask a question
- some useful equations are provided on the next page
- please show your work, results that are provided without derivation will be discounted

Points for each problem:

- Problem 1: *15 points*
- Problem 2: *25 points*
- Problem 3: *10 points*

Useful equations

- Euler-Lagrange equations (for problems 1 and 2):

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0\,. \tag{0.1}$$

- Principal moments of inertia (for problem 3):

$$\mathrm{I_{xx}} = \int \mathrm{dm}\,(y^2 + z^2)\,, \tag{0.2}$$

$$\mathrm{I_{yy}} = \int \mathrm{dm}\,(x^2 + z^2)\,, \tag{0.3}$$

$$\mathrm{I_{zz}} = \int \mathrm{dm}\,(x^2 + y^2)\,. \tag{0.4}$$

- Volume element in polar coordinates, ρ, ϕ, z, (for problem 3):

$$dV = \rho\, d\rho\, d\phi\, dz\,, \tag{0.5}$$

where $x = \rho\cos\phi$, $y = \rho\sin\phi$.

- Integrals of sinusoidal functions (for problem 3):

$$\int_0^{2\pi} \sin^2\phi\; d\phi = \pi\,. \tag{0.6}$$

(0.7)

- Volume of a cone of height h and base radius R (for problem 3):

$$V = \frac{\pi R^2 h}{3}\,. \tag{0.8}$$

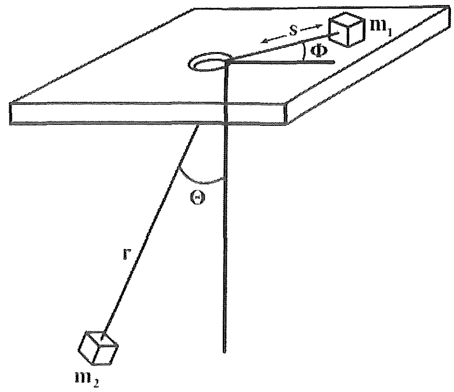

Figure 1.

Problem 1

Consider two masses m_1 and m_2 as in Fig. 1, connected by a string of length ℓ. The first mass (m_1) moves on a horizontal plane (no friction). The second mass (m_2) moves in a vertical plane. Compute the Lagrangian for the system and obtain the equations of motion for the two masses.
Note: for clarity, please don't forget to define your coordinates of choice.
[15 points]

Problem 2

Consider a sphere of radius a and moment of inertia I that rolls without slipping on the surface of a cylinder of radius R as in Fig. 2.

(1a) Compute the Lagrangian for the sphere using the variables shown in Fig. 2 (r, Θ, Φ). You do not need to specify what the moment of inertia is, you can leave it as I throughout the calculations.
[7 points]

(1b) Write the constraint equations (i.e. the equations relating r, Θ and Φ);
[8 points]

(1c) Using your results in (1a) and (1b), along with Euler-Lagrange equations, obtain the following equation:

$$\dot{\Theta}^2 = \frac{10}{7}\frac{g}{(R+a)}(1-\cos\Theta)\ , \qquad (0.9)$$

where $\dot{\theta} \equiv \frac{d\Theta}{dt}$. *Hint: in order to obtain the final result, you can use the following trick to perform the necessary integral:* $\ddot{\Theta} = \dot{\Theta}\frac{d\dot{\Theta}}{d\Theta}$.
[10 points]

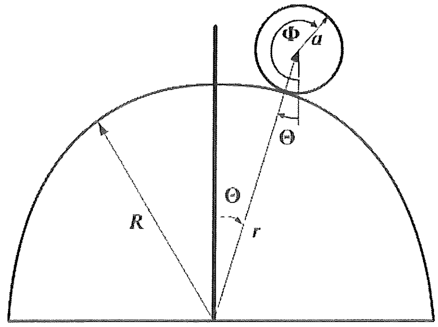

Figure 2. Notice: in the figure we have $r = R + a$.

Problem 3

Consider the cone in Fig. 3 (uniform density ρ_0, mass M, height h, base radius R). Compute the principal moments of inertia, I_{xx}, I_{yy} and I_{zz} (please refer to Fig. 3 and to the equations provided in the formula sheet).
[10 points]

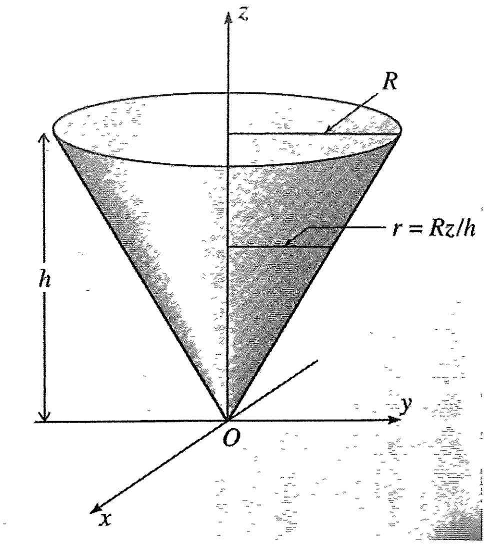

Figure 3.